THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 Stochastic Processes, 2020-21 Term 2

Take-home Midterm Test

Time and Date: 10:00am March 19 to 10:00am March 20

Answer all questions in both Part I and Part II (Total points: 120). Give adequate explanation and justification for all your computations and observations, and write your proofs in a clear and rigorous way.

Part I (100 points). Computations.

1. *(15 points)* Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $S = \{a, b, c\}$, transition matrix

$$P = \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix} \end{array},$$

and the initial distribution $\pi = (\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$. Compute the following

(a) $P_a(X_1 = b, X_2 = b, X_3 = b, X_4 = a, X_5 = c)$,

(b) $P_c(X_1 = a, X_2 = c, X_3 = c, X_4 = a, X_5 = b)$,

(c) $P_a(X_1 = b, X_3 = a, X_4 = c, X_6 = b)$,

(d) $P(X_1 = b, X_2 = b, X_3 = a)$,

(e) $P(X_2 = b, X_5 = b, X_6 = b)$.

2. *(15 points)* Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $S = \{x, y, z, w\}$ and transition matrix

$$P = \begin{array}{c} \begin{matrix} x & y & z & w \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{bmatrix} \end{array}.$$

(a) Compute $P(X_5 = z, X_6 = x, X_7 = z, X_8 = z | X_4 = y)$.

(b) Compute $E(f(X_5)f(X_6)|X_4 = w)$ for the function f with values $2, 3, 7$ and 3 at x, y, z and w respectively.

(c) For each $i, j \in S$, find ρ_{ij}, the probability that starting at i the chain ever visits j in finite time.

3. *(10 points)* Consider a Markov chain with state space $S = \{1,2,3\}$ and the transition matrix

$$P = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{3}{5} & \frac{1}{15} \end{bmatrix} \end{array}.$$

(a) For each $i = 1,2,3$ and all $k = 1,2,\cdots$, compute the probabilities that starting at i, the first visit to 3 occurs at time k.

(b) For each $i = 1,2,3$, find the probability that starting at i, the chain never visits 3 at any positive time.

4. *(10 points)* Consider the Markov chain with state space $S = \{1,2,\cdots,10\}$ and transition matrix

$$P = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \end{array}.$$

(a) Draw the transition graph.

(b) State the decomposition of state space by finding all the irreducible closed sets of recurrent states as well as the set of transient states.

(c) Write down the canonical form of transition matrix by reordering states.

5. *(15 points)* Given a branching process with the offspring distribution

$$p_0 = 0.5,\ p_1 = 0.1,\ p_3 = 0.4.$$

(a) Determine the extinction probability ρ.

(b) Let $X_0 = 1$. What is the probability that the population is extinct in the second generation ($X_2 = 0$), given that it did not die out in the first generation ($X_1 > 0$)?

(c) Still let $X_0 = 1$. What is the probability that the population is extinct in the third generation, given that it was not extinct in the second generation?

6. *(15 points)* Let X_n, $n \geq 0$, denote the capital of a gambler at the end of the nth play. His strategy is as follows. If his capital is 4 dollars or more, then he bets 2 dollars which earn him $4, 3$ or 0 dollars with respective probabilities 0.25, 0.30 and 0.45. If his capital is $1, 2$ or 3 dollars, then he plays more conservatively, bets 1 dollar, and this earns him either 2 or 0 dollars with respective probabilities 0.45 and 0.55. When his capital becomes 0, he stops.

   (a) Let Y_{n+1} be the net earnings at the $(n+1)$th play, that is,

   $$X_{n+1} = X_n + Y_{n+1}.$$

   Compute

   $$P(Y_{n+1} = k | X_n = i), \quad i = 0, 1, \cdots ; k = -2, -1, 0, 1, \cdots .$$

   (b) Explain that $\{X_n\}_{n \geq 0}$ is a Markov chain.

   (c) Compute the transition probabilities for the chain.

   (d) Classify the states, either recurrent or transient.

7. *(20 points)* Let $\{X_n\}_{n \geq 0}$ be a Markov chain over $S = \{1, 2, \cdots, 7\}$ with the following transition matrix

$$P = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.3 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

   Determine the limit $\lim_{n \to \infty} P^n(x, y)$ for any $x, y \in S$.

Part II (20 points) Theories and Applications.

8. *(10 points)* Let $\{X_n\}_{n \geq 0}$ be a stochastic process taking values in a countable state space S. Suppose there exists an integer $K \geq 1$ such that

$$P(X_n = i_n | X_0 = i_0, \cdots, X_{n-1} = i_{n-1}) = P(X_n = i_n | X_{n-K} = i_{n-K}, \cdots, X_{n-1} = i_{n-1})$$

   for all $i_\ell \in S$ with $0 \leq \ell \leq n$ and for all $n \geq K$. In other words, given all the past, the future depends only on the last K values. Such a process is called a K-dependent chain. For $K = 1$, we have the ordinary Markov chains. Their theory can, however, be reduced to that of the ordinary Markov chains by the following procedure.

   For each $n \geq 0$, let

$$Y_n = (X_n, X_{n+1}, \cdots, X_{n+K-1}).$$

   Then $\{Y_n\}_{n \geq 0}$ is a stochastic process taking values in the countable set $F = S^K = S \times \cdots \times S$. **Explain** that $\{Y_n\}_{n \geq 0}$ is an ordinary Markov chain.

9. *(10 points)* Let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain on the state space $S = \{1, \cdots, N\}$.

(a) Show that there exist $0 < C < \infty$ and $0 < \rho < 1$ such that for any states i, j,

$$P(X_m \neq j, m = 0, \cdots, n | X_0 = i) \leq C\rho^n, \quad \forall\, n.$$

(*Hint:* There exists a $\delta > 0$ such that for all i, the probability of reaching j some time in the first N steps, starting at i, is greater than δ. Why?)

(b) Show that (a) further implies $E(T_j) < \infty$, where T_j is the hitting time of j.

—THE END—

Solution.

1 (a).

$$\begin{aligned}P_a(X_1 = b, X_2 = b, X_3 = b, X_4 = a, X_5 = c) &= P(a,b)P(b,b)P(b,b)P(b,a)P(a,c)\\ &= \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{2}{3}\\ &= \frac{1}{32}.\end{aligned}$$

(b).

$$\begin{aligned}P_c(X_1 = a, X_2 = c, X_3 = c, X_4 = a, X_5 = b) &= P(c,a)P(a,c)P(c,c)P(c,a)P(a,b)\\ &= \frac{8}{375}.\end{aligned}$$

(c). Note that

$$P^2 = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} \frac{7}{20} & \frac{1}{4} & \frac{2}{5} \\ \frac{3}{16} & \frac{31}{48} & \frac{1}{6} \\ \frac{6}{25} & \frac{2}{15} & \frac{47}{75} \end{bmatrix} \end{array}.$$

$$\begin{aligned}P_a(X_1 = b, X_3 = a, X_4 = c, X_6 = b) &= P(a,b)P^2(b,a)P(a,c)P^2(c,b)\\ &= \frac{1}{180}.\end{aligned}$$

(d).

$$\begin{aligned}P(X_1 = b, X_2 = b, X_3 = a) &= (\pi(a)P(a,b) + \pi(b)P(b,b) + \pi(c)P(c,b))P(b,b)P(b,a)\\ &= \frac{17}{320}.\end{aligned}$$

(e). Note

$$P^3 = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} \frac{89}{400} & \frac{73}{240} & \frac{71}{150} \\ \frac{73}{320} & \frac{35}{64} & \frac{9}{40} \\ \frac{71}{250} & \frac{9}{50} & \frac{67}{125} \end{bmatrix} \end{array}.$$

$$\begin{aligned}P(X_2 = b, X_5 = b, X_6 = b) &= (\pi(a)P^2(a,b) + \pi(b)P^2(b,b) + \pi(c)P^2(c,b))P^3(b,b)P(b,b)\\ &= \frac{2373}{20480}.\end{aligned}$$

2 (a).

$$\begin{aligned} P(X_5 = z, X_6 = x, X_7 = z, X_8 = z | X_4 = y) &= P(y,z)P(z,x)P(x,z)P(z,z) \\ &= \frac{12}{125} = 0.096 \end{aligned}$$

(b).

$$\begin{aligned} \mathbb{E}[f(X_5)f(X_6)|X_4 = w] = &P(w,x)P(x,z)f(x)f(z) + P(w,y)P(y,y)f(y)f(y) + \\ &P(w,y)P(y,z)f(y)f(z) + P(w,w)P(w,x)f(w)f(x) + \\ &P(w,w)P(w,y)f(w)f(y) + P(w,w)P(w,w)f(w)f(w) \\ = &\frac{593}{50} = 11.86 \end{aligned}$$

(c). Note that $\{x, z\}$ is an irreducible closed set. Thus,

$$[\rho_{ij}] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ * & * & * & * \\ 1 & 0 & 1 & 0 \\ * & * & * & * \end{pmatrix},$$

where $*$'s are yet to be determined. By the one-step formulae,

$$\begin{aligned} \rho_{yw} &= P(y,w) + P(y,x)\rho_{xw} + P(y,y)\rho_{yw} + P(y,z)\rho_{zw} \\ \rho_{ww} &= P(w,w) + P(w,x)\rho_{xw} + P(w,y)\rho_{yw} + P(w,z)\rho_{zw}. \end{aligned}$$

Solve it, we have $\rho_{yw} = 0$ and thus $\rho_{ww} = P(w,w) = 0.5$, i.e.

$$[\rho_{ij}] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ * & * & * & 0 \\ 1 & 0 & 1 & 0 \\ * & * & * & 0.5 \end{pmatrix}.$$

Similarly, we consider

$$\begin{aligned} \rho_{yx} &= P(y,x) + \sum_{s \neq x} P(y,s)\rho_{sx} \\ \rho_{yy} &= P(y,y) + \sum_{s \neq y} P(y,s)\rho_{sy} \\ \rho_{yz} &= P(y,z) + \sum_{s \neq z} P(y,s)\rho_{sz}, \end{aligned}$$

and get $\rho_{yx} = 1$, $\rho_{yy} = 0.4$ and $\rho_{yz} = 1$. Finally, by considering similar one-step formulae with respect to w, we get $\rho_{wx} = 1$, $\rho_{wy} = 0.6$, $\rho_{wz} = 1$ and $\rho_{ww} = 0.5$, i.e.

$$[\rho_{ij}] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0.4 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0.6 & 1 & 0.5 \end{pmatrix}.$$

3 (a). By direct observation, one know that we can never visit to 3 if we start at state 1. Hence, $P_1(T_3 = k) = 0$ for all $k = 1, 2, \ldots$.

If we start at state 2, to have first visit to 3, we must never visit to state 1 (it is like a black hole that nothing can escape) and never go to state 3 before time k, i.e. we must stay at state 2 for $t = 1, 2, \ldots, k-1$.

Hence, $P_2(T_3 = k) = P_2(X_1 = 2, X_2 = 2, \ldots, X_{k-1} = 2, X_k = 3) = \dfrac{1}{3 \times 6^{k-1}}$

If we start at state 3, then $P_3(T_3 = 1) = P(3,3) = \dfrac{1}{15}$. For time $t = k > 1$, by similar reasoning as we start at state 2, we must first go to state 2 and then stay at state 2 until time k. Finally, we visit back to state 3.

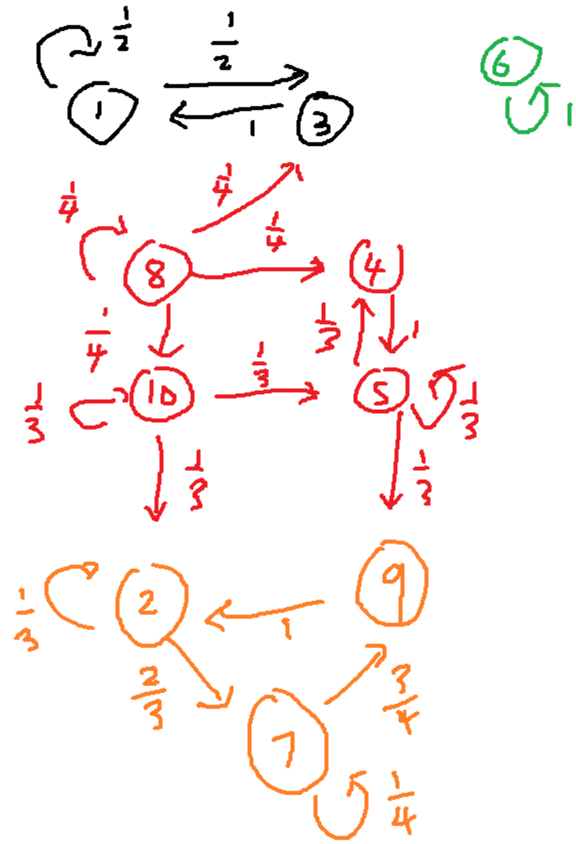
Hence, $P_3(T_3 = k) = P_3(X_1 = 2, X_2 = 2, \ldots, X_{k-1} = 2, X_k = 3) = \dfrac{3}{5} \times \dfrac{1}{6^{k-2}} \times \dfrac{1}{3} = \dfrac{1}{5 \times 6^{k-2}}$, for $k > 1$.

(b). Obviously, $P_1(T_3 = \infty) = 1$.

$$
\begin{aligned}
P_2(T_3 = \infty) &= 1 - \sum_{k=1}^{\infty} P_2(T_3 = k) \\
&= 1 - \sum_{k=1}^{\infty} \frac{1}{3 \times 6^{k-1}} \\
&= 1 - \frac{1}{3} \times \frac{1}{1 - \frac{1}{6}} \\
&= \frac{3}{5}.
\end{aligned}
$$

$$
\begin{aligned}
P_3(T_3 = \infty) &= 1 - \sum_{k=1}^{\infty} P_3(T_3 = k) \\
&= 1 - \frac{1}{15} - \sum_{k=2}^{\infty} \frac{1}{5 \times 6^{k-2}} \\
&= 1 - \frac{1}{15} - \frac{1}{5} \times \frac{1}{1 - \frac{1}{6}} \\
&= \frac{52}{75}.
\end{aligned}
$$

4 (a).



(b). The irreducible closed sets of recurrent states are $\mathcal{C}_1 = \{1, 3\}, \mathcal{C}_2 = \{2, 7, 9\}$ and $\mathcal{C}_3 = \{6\}$. The set of transient state is $\mathcal{S}_T = \{4, 5, 8, 10\}$.

(c).

$$\bar{P} = \begin{array}{c} \begin{array}{cccccccccc} 1 & 3 & 2 & 7 & 9 & 6 & 4 & 5 & 8 & 10 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{array}.$$

5 (a). The mean $\mu = 0.1 \times 1 + 0.4 \times 3 = 1.3 > 1$. Hence, the extinction probability $\rho \in [0, 1)$. Now to find ρ, we need to solve $\Phi(t) = \sum_{k=0}^{\infty} p_k t^k = t$, i.e.

$$\begin{aligned} t &= 0.5 + 0.1t + 0.4t^3 \\ 4^3 - 9t + 5 &= 0 \\ (t-1)(4t^2 + 4t - 5) &= 0 \end{aligned}$$

Solve it, we have $t = 1$, $\frac{\sqrt{6}-1}{2}$ or $\frac{-\sqrt{6}-1}{2}$. Since $\rho \in [0, 1)$, we must have $\rho = \frac{\sqrt{6}-1}{2}$.

(b).

$$\begin{aligned} P_1(X_2 = 0 | X_1 > 0) &= \frac{P_1(X_2 = 0, X_1 > 0)}{P_1(X_1 > 0)} \\ &= \frac{P_1(X_1 = 1, X_2 = 0) + P_1(X_1 = 3, X_2 = 0)}{1 - P_1(X_1 = 0)} \\ &= \frac{0.1 \times 0.5 + 0.4 \times 0.5^3}{1 - 0.5} \\ &= 0.2 \end{aligned}$$

(c). Note that $P_1(X_2 = 0) = P_1(X_1 = 0) + P_1(X_1 > 0, X_2 = 0) = 0.5 + 0.5 \times 0.2 = 0.6$. Therefore,

$$\begin{aligned} P_1(X_3 = 0) &= P_1(X_1 = 0) + P_1(X_1 = 1)P(X_3 = 0 | X_1 = 1) + P_1(X_1 = 3)P(X_3 = 0 | X_1 = 3) \\ &= 0.5 + 0.1 \times 0.6 + 0.4 \times 0.6^3 \\ &= 0.6464 \end{aligned}$$

$$\begin{aligned} P_1(X_3 = 0 | X_2 > 0) &= \frac{P_1(X_3 = 0, X_2 > 0)}{P_1(X_2 > 0)} \\ &= \frac{P_1(X_3 = 0) - P(X_2 = 0)}{1 - P_1(X_2 = 0)} \\ &= \frac{0.6464 - 0.6}{1 - 0.6} \\ &= 0.116 \end{aligned}$$

6 (a). For $i = 0$,

$$P(Y_{n+1} = k | X_n = 0) = \begin{cases} 1, & \text{if k = 0} \\ 0, & \text{otherwise} \end{cases}$$

For $1 \le i \le 3$

$$P(Y_{n+1} = k | X_n = i) = \begin{cases} 0.45 & \text{if k = 1} \\ 0.55, & \text{if k = -1} \\ 0, & \text{otherwise} \end{cases}$$

For $i \ge 4$

$$P(Y_{n+1} = k | X_n = i) = \begin{cases} 0.25 & \text{if k = 2} \\ 0.30, & \text{if k = 1} \\ 0.45, & \text{if k = -2} \\ 0, & \text{otherwise} \end{cases}$$

(b). It is obvious that

$$\begin{aligned} P(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_1 = x_1) &= P(Y_{n+1} = x_{n+1} - x_n | X_n = x_n, \ldots, X_1 = x_1) \\ &= P(Y_{n+1} = x_{n+1} - x_n | X_n = x_n) \\ &= P(X_{n+1} = x_{n+1} | X_n = x_n) \end{aligned}$$

is independent of previous states $X_0, X_1, \ldots, X_{n-1}$. Hence, $\{X_n\}_{n \ge 0}$ is a Markov chain.

(c).

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0.55 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0.55 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0.55 & 0 & 0.45 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0.45 & 0 & 0 & 0.3 & 0.25 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0.45 & 0 & 0 & 0.3 & 0.25 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0.3 & 0.25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(d). State 0 is recurrent since $\rho_{00} \ge P(0,0) = 1$. Other states are transient since for $x = 1, 2$ or 3, one can go from $x \to (x-1) \to \cdots \to 0$ with probability > 0, i.e. a positive probability that goes to the absorbing state 0 without hitting itself. Similarly, for $x \ge 4$, there is a path $x \to (x-2) \to (x-4) \to \cdots \to (4 + (x \mod 2)) \to (2 + (x \mod 2)) \to (1 + (x \mod 2)) \to (x \mod 2) \cdots \to 0$ with probability > 0.

7. First note that $\mathcal{C}_1 = \{1,5\}$ and $\mathcal{C}_2\{4,6,7\}$ are irreducible closed set and $S_T = \{2,3\}$ is the set of transient state. By reordering the index, we have

$$\bar{P} = \begin{array}{c} \begin{matrix} 1 & 5 & 4 & 6 & 7 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0.4 & 0 & 0 & 0.2 & 0.3 \\ 0 & 0.2 & 0.3 & 0 & 0 & 0 & 0.5 \end{bmatrix} \end{array}.$$

First, We solve $\lim_{k\to\infty} \bar{P}^k$. Now, we need to solve $\pi_1 P_{\mathcal{C}_1} = \pi_1$ under the condition $\sum_{x\in\mathcal{C}_1} \pi_1(x) = 1$ and $\pi_1(x) \geq 0$ for all $x \in \mathcal{C}_1$, i.e.

$$\begin{cases} \pi_1(1) & = 0.7\pi_1(1) + 0.6\pi_1(5) \\ \pi_1(5) & = 0.3\pi_1(1) + 0.4\pi_1(5) \\ \pi_1(1) + \pi_1(5) & = 1 \\ \pi_1(x) & \geq 0 \quad \forall x \in \mathcal{C}_1, \end{cases}$$

and get $\pi_1 = (2/3, 1/3)$.

Similarly, we have $\pi_2 P_{\mathcal{C}_2} = \pi_2$ under the condition $\sum_{x\in\mathcal{C}_2} \pi_2(x) = 2$ and $\pi_2(x) \geq 0$ for all $x \in \mathcal{C}_2$, i.e.

$$\begin{cases} \pi_2(4) & = 0.5\pi_2(4) + \pi_2(7) \\ \pi_2(6) & = 0.5\pi_2(4) + 0.2\pi_2(6) \\ \pi_2(7) & = 0.8\pi_2(6) \\ \pi_2(4) + \pi_2(6) + \pi_2(7) & = 1 \\ \pi_2(x) & \geq 0 \quad \forall x \in \mathcal{C}_2, \end{cases}$$

and get $\pi_2 = (8/17, 5/17, 4/17)$.

We further solve

$$\begin{cases} \rho_{\mathcal{C}_1}(2) & = 0.1 + 0.2\rho_{\mathcal{C}_1}(2) + 0.3\rho_{\mathcal{C}_1}(3) \\ \rho_{\mathcal{C}_1}(3) & = 0.2 + 0.5\rho_{\mathcal{C}_1}(3) \end{cases}$$

and

$$\begin{cases} \rho_{\mathcal{C}_2}(2) & = 0.4 + 0.2\rho_{\mathcal{C}_2}(2) + 0.3\rho_{\mathcal{C}_2}(3) \\ \rho_{\mathcal{C}_2}(3) & = 0.3 + 0.5\rho_{\mathcal{C}_2}(3) \end{cases}$$

and get $\rho_{\mathcal{C}_1}(2) = 0.275, \rho_{\mathcal{C}_1}(3) = 0.4, \rho_{\mathcal{C}_2}(2) = 0.725$ and $\rho_{\mathcal{C}_2}(3) = 0.6$.

By what we have learnt in lecture, we have

$$\lim_{n\to\infty} \bar{P}^n = \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & 2 & 3 \end{array} \\ \begin{bmatrix} \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_1 \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} \pi_2 \\ \vdots \\ \pi_2 \end{bmatrix} & 0 & 0 \\ \rho_{\mathcal{C}_1}(2)\pi_1 & \rho_{\mathcal{C}_2}(2)\pi_2 & 0 & 0 \\ \rho_{\mathcal{C}_1}(3)\pi_1 & \rho_{\mathcal{C}_2}(3)\pi_2 & 0 & 0 \end{bmatrix} \end{array}. = \begin{array}{c} \begin{array}{ccccccc} 1 & 5 & 4 & 6 & 7 & 2 & 3 \end{array} \\ \begin{bmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8/17 & 5/17 & 4/17 & 0 & 0 \\ 0 & 0 & 8/17 & 5/17 & 4/17 & 0 & 0 \\ 0 & 0 & 8/17 & 5/17 & 4/17 & 0 & 0 \\ 11/60 & 11/120 & 29/85 & 29/136 & 29/170 & 0 & 0 \\ 4/15 & 2/15 & 24/85 & 3/17 & 12/85 & 0 & 0 \end{bmatrix} \end{array}.$$

Hence,

$$\lim_{n\to\infty} P^n = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 2/3 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 11/60 & 0 & 0 & 29/85 & 11/120 & 29/136 & 29/170 \\ 4/15 & 0 & 0 & 24/85 & 2/15 & 3/17 & 12/85 \\ 0 & 0 & 0 & 8/17 & 0 & 5/17 & 4/17 \\ 2/3 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 8/17 & 0 & 5/17 & 4/17 \\ 0 & 0 & 0 & 8/17 & 0 & 5/17 & 4/17 \end{bmatrix} \end{array}.$$

8. Let $y_k = (i_{kk}, i_{k(k+1)}, \ldots i_{k(k+K-1)})$ for all $k \geq 0$. To have $P(Y_{n+1} = y_{n+1} | Y_0 = y_0, Y_1 = y_1, \ldots Y_n = y_n)$ meaningful, we must have $i_{km} = i_{k'm}$ for all m and $k, k' = 0, 1, \ldots, n+K-1$. We denote that common i_{km} by i_m and write $y_{n+1} = (j_{n+1}, j_{n+2}, \ldots, j_{n+K})$. In our setting, we have $j_l = i_l$ for $l = n+1, n+2, \ldots, (n+K-1)$. Thus, we have

$$\begin{aligned} & P(Y_{n+1} = y_{n+1} | Y_0 = y_0, Y_1 = y_1, \ldots, Y_n = y_n) \\ = & P(Y_{n+1} = y_{n+1} | X_0 = i_0, \ldots, X_{n+K-1} = i_{n+K-1}) \\ = & P((X_{n+1}, X_{n+2}, \ldots X_{n+K}) = (j_{n+1}, j_{n+2}, \ldots, j_{n+K}) | X_0 = i_0, \ldots, X_{n+K-1} = i_{n+K-1}) \\ = & P(X_{n+K} = j_{n+K} | X_0 = i_0, \ldots X_{n+K-1} = i_{n+K-1}) \\ = & P(X_{n+K} = j_{n+K} | X_n = i_n, X_{n+1} = i_{n+1} \ldots X_{n+K-1} = i_{n+K-1}) \\ = & P((X_{n+1}, X_{n+2}, \ldots X_{n+K}) = (j_{n+1}, j_{n+2}, \ldots, j_{n+K}) | X_n = i_n, X_{n+1} = i_{n+1} \ldots X_{n+K-1} = i_{n+K-1}) \\ = & P(Y_{n+1} = y_{n+1} | Y_n = y_n). \end{aligned}$$

Hence, $\{Y_n\}_{n\geq 0}$ is an ordinary Markov chain.

9. Given any state j, let $S_m(j) := \{k \in \mathcal{S} | P^{m_0}(k,j) > 0 \text{ for some } m_0 \in \{1,2,\dots m\}\}$. We claim that $S_N(j) = \mathcal{S}$. One can easily verify the following properties (i) $S_m(j) \subseteq S_{m+1}(j)$, (ii) if $S_m(j) = S_{m+1}(j)$, then $S_m(j) = S_{m+l}(j)$ for all $l \geq 0$ (Exercise!).

By the fact that X_n is irreducible, for any $k_1 \in \mathcal{S}$, there is some $N_{k_1} \in \mathbb{N}$ such that $P^{N_{k_1}}(k_1, j) > 0$. Let $N_0 = \max\{N_1, N_2, \dots, N_n\}$, we have $S_{N_0}(j) = \mathcal{S}$.

Next, we claim that there is some $k_0 \in \mathcal{S}$ such that $P(k_0, j) > 0$. Otherwise $P(k_0, j) = 0$ for all $k_0 \in \mathcal{S}$ and thus $P^m(k_0, j) = 0$ for all $k_0 \in \mathcal{S}$ and $m \in \mathbb{N}$, contradicting to the irreducibility of X_n. Hence, $|S_1(j)| \geq 1$. By our property (i), we have $|S_m(j)| \geq 1$ for all $m \in \mathbb{N}$.

We claim that $|S_N(j)| = N$. If not, then there is some $m_0 \in \{1,2,\dots N\}$ such that $1 \leq |S_N(j)| \leq m_0 - 1$. By pigeonhole principle (i.e. counting), there must be some m_1, m_2 such that $|S_{m_1}(j)| = |S_{m_2}(j)|$, say $m_1 < m_2$. Then, by property (i), we have $S_{m_1}(j) \subseteq S_{m_1+1}(j) \subseteq \cdots \subseteq S_{m_2}(j)$ and hence $S_{m_1}(j) = S_{m_1+1}(j) = \cdots = S_{m_2}(j)$. By property (ii), we have $|S_N(j)| \leq m_0 - 1 < N$ for all $m \in \mathbb{N}$, contradicting the fact that $|S_N(j)| = |\mathcal{S}| = N$. Hence, we conclude that $S_N(j) = \mathcal{S}$.

Hence, for any $i \in \mathcal{S}$, there exists some $m_{ij} \in \{1,2,\dots,N\}$ such that $P^{m_{ij}}(i,j) > 0$. Let $\delta = \min_{i,j=1,2,\dots n}\{P^{m_{ij}}(i,j)\}/2 > 0$. Then, $P_i(T_j \leq N) \geq \delta$ and $\delta \in (0,1)$.

Let $\rho = (1-\delta)^{1/N} \in (0,1), C = \rho^{-N}$. Then,

$$P(X_m \neq j, m = 0, \dots, n | X_0 = i) \leq 1 \leq C\rho^n$$

for $n = 0, 1, \dots, N$.

Next, suppose that

$$P(X_m \neq j, m = 0, \dots, n | X_0 = i) \leq C\rho^n$$

for $n = 1, 2, \dots, K$. for some positive integer K. Then, if $P(X_K = j | X_0 = i) = 1$, then $P(X_m \neq j, m = 0, \dots, K+N | X_0 = i) = 0 \leq C\rho^n$. Otherwise,

$$\begin{aligned}
&P(X_m \neq j, m = 0, \dots, K+N | X_0 = i) \\
&= P(X_m \neq j, m = K+1, \dots, K+N | X_K \neq j) P(X_m \neq j, m = 0, \dots, K | X_0 = i) \\
&\leq P(X_m \neq j, m = K+1, \dots, K+N | X_K \neq j) \times C\rho^K \\
&\leq (1-\delta) \times C\rho^K \\
&= C\rho^{N+K},
\end{aligned}$$

where the last inequality comes from the fact that wherever X_K is, it reaches state j within N steps with probability at least δ.

The conclusion now follows from induction.

(b).

$$\begin{aligned} E_i(T_j) &= \sum_{k=1}^{\infty} k P_i(T_j = k) \\ &\leq \sum_{k=0}^{\infty} (k+1) P_i(T_j \geq (k+1)) \\ &= \sum_{k=0}^{\infty} (k+1) P(X_m \neq j, m = 0, \ldots, k | X_0 = i) \\ &\leq C \sum_{k=0}^{\infty} (k+1) \rho^k \\ &= \frac{C}{(1-\rho)^2} < \infty \end{aligned}$$

Thus,

$$E(T_j) = \sum_{i \in \mathcal{S}} P(X_0 = i) E_i(T_j) \leq \sum_{i \in \mathcal{S}} P(X_0 = i) \frac{C}{(1-\rho)^2} = \frac{C}{(1-\rho)^2} < \infty.$$